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Fuzzy Boundary Element Method for the Analysis of Imprecisely Defined Systems

S. S. Rao* and Lingtao Cao†
University of Miami, Coral Gables, Florida 33124-0624

Many engineering systems contain uncertainties that cannot be described by either deterministic or probabilistic approaches. The uncertainties present may be associated with parameters that are vague, imprecise, or linguistic. For example, the geometry, material properties, external actions (loads), and boundary conditions may be imprecise in a practical system. A fuzzy boundary element method for the analysis of imprecisely defined systems is developed. Starting from the basic concepts of fuzzy sets and fuzzy arithmetic, the various steps of the boundary element method are redefined using fuzzy concepts. The resulting fuzzy equations are solved using a fuzzified version of Gaussian elimination procedure coupled with truncation. The truncation method limits the growth of interval ranges of response parameters to obtain realistic and accurate solutions. The procedure is illustrated by considering the analysis of a potential flow problem. The methodology is applicable to uncertain systems that are described in linguistic terms as well as those that are described by incomplete information. The approach represents a unique methodology that enables the analysis and design of many engineering systems more realistically.

Introduction

IN the face of global competition, engineers are expected to analyze, design, and manufacture systems that are reliable and economical. This requires that accurate and realistic analysis and design methods be available to predict the performance of the system. The traditional analysis approaches, which assume system parameters to be completely deterministic and precise, are not valid in many cases. Most practical systems involve parameters that are described by vague, imprecise, and fuzzy statements.

The probabilistic and reliability analysis techniques assume that the uncertain parameters, such as stress and strength of a mechanical system, are described as random variables with known probability distribution functions. The evaluation of an event, such as the one involved in the computation of the reliability of the system, requires the integration of the probability density functions over appropriate regions. If the system parameters are described as fuzzy quantities, probabilistic methods are not applicable, and one has to develop suitable fuzzy computational procedures.

In a mechanical or structural system, fuzziness or imprecision can be present in the geometry, material properties, applied loads, and boundary conditions. The geometric parameters are imprecise due to manufacturing tolerances, imperfect/inaccurate machine settings during production, and human error. The material properties are imprecise due to uncertain material composition, uncontrollable manufacturing conditions such as annealing/quenching time and temperature, as well as unknown environmental conditions of use such as corrosion, temperature, and humidity. The applied loads are imprecise due to unpredictable external conditions such as earthquakes, wind velocities, and operating dynamic conditions involved.

In some engineering problems, the boundary conditions may not be known precisely. For example, the exact support conditions of the gudgeon pin in a piston, the clevis pin in a knuckle joint, or the arbor in a milling machine are not known. The bolts between a machine and its foundation may not have been tightened properly. The true boundary conditions in all of these cases lie somewhere between simple supports and fixed ends. In the case of stress analysis, let the elastic region be bounded by a closed curve. The boundary S is composed of S_u and S_t, where S_u and S_t are parts of the boundary on which the displacements and tractions are specified, respectively. Let AB be a crack, BC a fixed edge, AE and CD free edges, and ED the edge on which load (traction) is applied (Fig. 1). Thus, the boundary $BAEDC$ denotes S_t, and BC indicates S_u. The degree of fixation along BC, the length of BC, as well as the magnitude of the load may have to be treated fuzzy.

The boundary element method has been established as a practical problem-solving tool and is effectively incorporated into digital computer algorithms for the analysis of complex engineering problems. The method was developed essentially from boundary integral equations. This paper presents a new methodology, based on fuzzy set theory and boundary element method, for the analysis of engineering systems involving imprecisely and vaguely defined parameters.

Brief Literature Review

Stochastic Boundary Element Method

Although there is no suitable technique available for the analysis of all types of imprecision, the stochastic boundary element method can be used to handle uncertain parameters that are described by probability distributions. Although the stochastic boundary element method was not developed to the same extent as stochastic finite element method,[1] several investigations have been made in the past several years. Nakagiri et al.[2] presented a stochastic boundary element approach for stress analysis problems.[2] The application of boundary element method for the solution of random heat conduction and general potential problems was presented.[3–5] The boundary element solution of groundwater flow problems was considered by Serrano and Unny,[6] Cheng and Lafe,[7] Cheng et al.,[8] and Kaljevic and Saigal.[9] The boundary integral/element methods for the two- and three-dimensional thermal and hydroacoustic problems with random media were presented by Manolis and Shaw.[10,11] The dynamic analysis of stochastic media by boundary elements was considered by Callerio et al.[12]

Lafe and Cheng[13] presented a global-interpolation-function-based boundary element method for deterministic, nonhomogeneous and stochastic flows in porous media. A boundary element method for stochastic flow problems in a semiconfined aquifer with random boundary conditions was presented by Zhu and Satish in Ref 14. Other investigations on stochastic boundary element method include those of Breitung et al.[15] and Roy and Grilli.[16]

Fuzzy Approach

Although fuzzy set theory was developed in 1965,[17] not many applications resulted for several years. However, recent applications of fuzzy set theory to scientific areas such as artificial intelligence, image processing, speech recognition, biological and medical sciences, decision theory, economics, geography, sociology, psychology,

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*Professor and Chairman, Department of Mechanical Engineering. Associate Fellow AIAA.

†Graduate Student, Department of Mechanical Engineering.

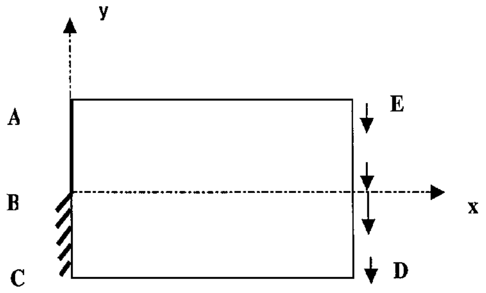

Fig. 1 Boundary conditions.

linguistics, and semiotics indicate that fuzzy set theory may not be a theory in search of applications but indeed a useful tool for the quantification of impreciseness and vagueness present in many real-life problems. Most engineering applications of fuzzy set theory have been in the areas of controls, decision making, and optimization. Rao[18,19] presented a fuzzy approach for the description and optimum design of imprecisely defined mechanical and structural systems involving single and multiple objectives. Rao and Sawyer[20] presented a fuzzy finite element method for the modeling and analysis of uncertain engineering systems.

Fuzzy Analysis

Fuzzy Description

In any design problem, it is difficult to initially fix the values of the variables involved. Similarly, the conclusion of a design process may not be expressible as a unique result. Actually, it is desirable to obtain a range of data to be able to judge the satisfaction levels of different design criteria to find the best manufacturing strategy. This implies that, in most cases, we may not be able to describe a problem using precise numbers or define a design domain with crisp boundary. The problem is stated in a linguistic manner with imprecise information. Usually, each imprecise variable is assigned a range of values, which leads to a range of design performance.

A linguistic statement can be viewed as a fuzzy set that displays the distribution of a parameter. The distribution of a linguistic variable can be described using a suitable function, called a membership function, whose value ranges from 0 to 1. If the membership value is equal to 0 for a specified value of the parameter, it implies that the value is definitely outside the fuzzy set. Similarly, if the membership value is equal to 1 for some parameter, it means that the value is definitely a member of the fuzzy set. Those values with membership between 0 and 1 are defined vaguely. From a design point of view, the value of membership function changes from 0 to 1, where 0 denotes the boundary of design permission (border of design prohibition) and 1 denotes the precise or crisp design character.

Fuzzy Numbers

To present the concept of a fuzzy number mathematically, let X be a crisp set of objects, called the universe, whose generic elements are denoted x. The membership for a crisp subset F of X can be viewed as a characteristic function μ_F with

$$\mu_F(x) = \begin{cases} 1 & \text{if} \quad x \in F \\ 0 & \text{if} \quad x \notin F \end{cases} \tag{1}$$

The set {0, 1} is known as a valuation set, which represents a gradual change of membership degree rather than abrupt transition from membership to nonmembership. The set F is taken as a fuzzy set if the valuation set is allowed to be the real interval [0, 1]. A fuzzy set F is characterized by the set of pairs

$$F = \{[x, \mu_F(x)], x \in X\}, \qquad 0 \le \mu_F(x) \le 1 \tag{2}$$

where $\mu_F(x)$ is the grade of membership function or degree of compatibility of x to F. The more $\mu_F(x)$ is closer to 1, the more x will belong to F. The membership function reflects the degree of belongingness that can be constructed on the basis of statistical data. For example, in some cases, the probability density function can be applied to describe the membership function[21]

$$\mu(x) = p(x)/\max[p(x)] \tag{3}$$

where $p(x)$ is the probability density function. Note that not all fuzzy sets have statistical basis for the definition of their membership functions. Although most of the practical fuzzy quantities have nonlinear membership functions, linear membership functions are commonly used for simplicity in numerical computations.

Fuzzy Arithmetic

The operation between two fuzzy sets A and B can be written as[22]

$$\mu_{C=A*B}(z) = \underset{z=x*y}{V} [\mu_A(x) \wedge \mu_B(y)], \qquad \forall x, y, z \in R \tag{4}$$

where $*$ denotes an arithmetic operation such as $+, -, \times, \div$. It denotes a combination of max–min operations performed on the membership values of the elements of fuzzy set. Note that the number of elements in the fuzzy set C resulting from a series of fuzzy operations will become quite large if the max–min convention is used in each operation. Thus, there is a need to find a convenient way to express fuzzy numbers and their operations for a realistic prediction of the performance of a practical system.

α-Level Cuts for a Fuzzy Number

An interval expression is a convenient way to express fuzzy numbers as sets of upper and lower bounds of a finite number of α-cut subsets. Defined by different grades of compatibility to a design specification, these design values in a fuzzy set can be grouped further in the expression for interval numbers. Each level has a membership value that represents the degree of design constraint satisfaction. If F is a fuzzy number with its membership function μ_F determined through a series of fuzzy operations, its α-cut subsets are defined as

$$F_\alpha = \{x \in X, \mu_F(x) \ge \alpha\}, \qquad 0 \le \alpha \le 1 \tag{5}$$

For the ith α cut, the upper and lower bounds (f^L and f^U) are given by

$$f^L = \min\{f : f \in F_\alpha\} \tag{6}$$

$$f^u = \max\{f : f \in F_\alpha\} \tag{7}$$

For a given variable, if n sets of α level are established for all fuzzy quantities of the variable, a 2 by n array can be used to represent all of these quantities as follows:

$$F = \left\{(f^L, f^U)_1, (f^L, f^U)_2, \ldots, (f^L, f^U)_n\right\} \tag{8}$$

In addition, each fuzzy number possesses two attributes, convexity and normality. Convexity requires that the shape of the membership function be convex, whereas normality implies that the highest grade of membership function is equal to 1.

In this work, the ranges of parameters are assumed to be distributed linearly. Given three bounding values of F, we can obtain the two interval values at any specified α level, $\mu(F) = \alpha$, as (Fig. 2)

$$F_\alpha = (\underline{f}_\alpha, \bar{f}_\alpha) = [\alpha f_1 + (1-\alpha)\underline{f}_0, \alpha f_1 + (1-\alpha)\bar{f}_0] \tag{9}$$

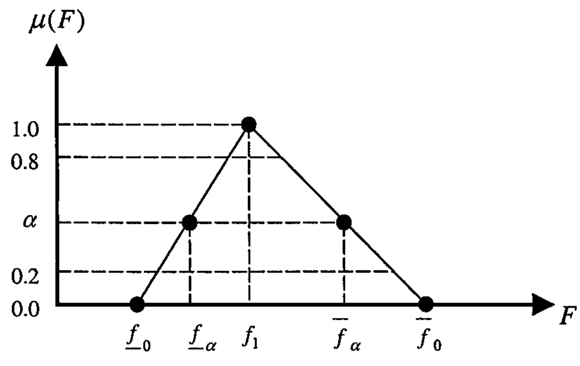

Fig. 2 Triangular fuzzy set.

When $\mu(F)=0$, then $F_0=(\underline{f}_0, \bar{f}_0)$, and when $\mu(F)=1$, then $F_1=f_1$. Once the fuzzy quantities are expressed in interval form, the fuzzy arithmetic operations can be carried using interval operations at each of the n α levels independently.

Interval Operations

The interval arithmetic operations on real numbers are defined as[23]

$$[\underline{a}, \bar{a}] * [\underline{b}, \bar{b}] = \{x * y|_{\underline{a} \le x \le \bar{a}, \underline{b} \le y \le \bar{b}}\} \tag{10}$$

where $*$ denotes an arithmetic operation sign, such as $+$, $-$, $\times$ (or $\cdot$), or $\div$ (or /). The division operation $[\underline{a}, \bar{a}]/[\underline{b}, \bar{b}]$ is defined only if $0 \notin [\underline{b}, \bar{b}]$. More explicitly, Eq. (10) can be expressed as

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}] \tag{11}$$

$$[\underline{a}, \bar{a}] - [\underline{b}, \bar{b}] = [\underline{a} - \bar{b}, \bar{a} - \underline{b}] \tag{12}$$

$$[\underline{a}, \bar{a}] \cdot [\underline{b}, \bar{b}] = [\min(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}), \max(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b})] \tag{13}$$

$$[\underline{a}, \bar{a}]/[\underline{b}, \bar{b}] = [\underline{a}, \bar{a}] \cdot [1/\bar{b}, 1/\underline{b}] \qquad \text{if} \qquad 0 \notin [\underline{b}, \bar{b}] \tag{14}$$

Note that a real number a is denoted by a degenerate interval $[a, a]$. Equation (10) indicates that interval addition and interval multiplication are both associative and commutative. Although the distributive law does not hold true always, the subdistributive law and the inclusion monotonicity law hold true. The interval arithmetic operations can be extended to matrix computations.

Manipulation of Linguistic Variables

As stated earlier, a problem can be described in a linguistic style: a statement assembled by words with a vague meaning associated with the words. Some labels such as "strong, flexible, tall, and large"; hedges such as "very, quite, greatly, and extremely"; negation such as "not, none, and neither"; and connectives such as "and, but, and or" can be composed together to define a complicated linguistic statement. However, a convert rule can be used to transform such statements into equivalent fuzzy representations.[24–26] If A and B denote two fuzzy sets used to describe the yield strength of a material, then some of the possible linguistic statements are indicated in Table 1.

Fuzzy Integration[27,28]

Consider a fuzzy function $\tilde{f}$ to be integrated over the crisp interval $[a, b]$. The function $\tilde{f}(x)$ can be considered to be a fuzzy number whose α-level curves are defined as $\mu_{\tilde{f}(x)}(y)=\alpha$ for all $\alpha \in [0, 1]$ and the parameters x and α are assumed to have two continuous solutions, $y=f_\alpha^+(x)$ and $y=f_\alpha^-(x)$ for $\alpha \neq 1$ and one solution for $\alpha=1$. The solutions $f_\alpha^+(x)$ and $f_\alpha^-(x)$ satisfy the relation

$$f_{\alpha_1}^+(x) \ge f_\alpha^+(x) \ge f(x) \ge f_\alpha^-(x) \ge f_{\alpha_1}^-(x) \qquad \text{for all } \alpha_1 \ge \alpha$$

Then the integral of $\tilde{f}(x)$ over $[a, b]$ is defined as

$$\tilde{I}(a,b) = \left\{\left(\int_a^b f_\alpha^-(x)\cdot \mathrm{d}x + \int_a^b f_\alpha^+(x)\cdot \mathrm{d}x\right), \alpha\right\} \tag{15}$$

which can be seen to be a fuzzy set in which the degree of membership α is assigned to the integral of any α-level curve of $\tilde{f}(x)$ over $[a, b]$. Equation (15) is consistent with the extension principle. Alternatively, if LR representation is used for fuzzy numbers, $\tilde{f}(x)$ is assumed to be

$$\tilde{f}(x) = [f(x), s(x), t(x)]_{LR} \tag{16}$$

for all $x \in [a, b]$, where the mean function $f(x)$ and the spread functions $s(x)$ and $t(x)$ are positive integral functions on $[a, b]$. According to Dubois and Prade,[29] the integral $\tilde{I}(a, b)$ can be represented as

$$\tilde{I}(a,b) = \left(\int_a^b f(x)\cdot \mathrm{d}x, \int_a^b s(x)\cdot \mathrm{d}x, \int_a^b t(x)\cdot \mathrm{d}x\right)_{LR} \tag{17}$$

Next, consider the integration of a crisp function $f(x)$ over a fuzzy domain (interval) bounded by two normalized convex fuzzy sets $\tilde{a}$ and $\tilde{b}$ with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ that can be interpreted as the degrees of confidence to which x can be considered as a lower and an upper bound of the fuzzy domain. If $\underline{a}_0$ and $\bar{b}_0$ denote the lower and upper limits of the supports of $\tilde{a}$ and $\tilde{b}$, respectively, the membership function of the integral $\tilde{I}$ can be represented as

$$\mu_{\tilde{I}}(z) = \sup_{x,y \in K} \min\left[\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\right] \tag{18}$$

where

$$z = \int_x^y f(x')\cdot \mathrm{d}x'$$

The described procedure can be extended to evaluate the fuzzy integral I of a fuzzy function over a fuzzy interval. In this case, the limits of integration are fuzzy numbers. For a given α level, there will be lower and upper bounds for the lower and upper limits of integration. Thus, there are four combinations of crisp intervals of integration and, hence, four combinations of crisp intervals of integration. To find the α-cut bounds $[I^L, I^R]$, the procedure for the integration of a fuzzy function over a crisp interval outlined earlier is used to evaluate the integral for each of the four possible crisp intervals of integration. The lower limit I^L is simply the minimum of these four values resulting from the integration of $f_\alpha^-(x)$. The upper bound I^R is the maximum of these four values resulting from the integration of $f_\alpha^+(x)$. For example, to evaluate the integral of a fuzzy function over a fuzzy interval at six α levels, a total of 48 deterministic integral operations would have to be performed.

Table 1 Fuzzy representation of typical linguistic statement

Set	Linguistic statement	X: 1	2	3	4	5	6
A	Low	0.10	0.30	0.50	0.70	0.90	1.00
$\bar{A}$	Not low	0.90	0.70	0.50	0.30	0.10	0.00
A^2	Very low	0.01	0.09	0.25	0.49	0.81	1.00
$\bar{A}^2$	Not very low	0.99	0.91	0.75	0.51	0.19	0.00
B	High	1.00	0.80	0.60	0.40	0.20	0.00
B^2	Very high	1.00	0.64	0.36	0.16	0.04	0.00
$A^2 \cup B^2$	Very low or very high	1.00	0.64	0.36	0.49	0.81	1.00
$A^2 \cup \bar{A}^2$	Low but not very low	0.10	0.30	0.50	0.51	0.19	0.00

Fuzzy Boundary Element Method (Fuzzy BEM)

In traditional boundary element methods, the boundary integral equation is expressed in matrix form by evaluating the integrals numerically to generate the coefficients (elements) of the boundary integral matrices. The resulting boundary integral equations can be expressed, by incorporating the boundary conditions, as a system of matrix equations. These equations are solved using a suitable technique such as Gaussian elimination or one of its variants. In the case of fuzzy boundary element method (BEM), the system parameters such as boundary conditions and geometry, material properties, as well as external conditions (applied loads), are assumed to be uncertain and imprecise. Thus, the boundary integrals involving fuzzy parameters are to be evaluated over fuzzy regions. The resulting system of equations are to be solved using methods based on fuzzy arithmetic. All of these theoretical and computational aspects are discussed in the following section.

Basic Procedure

Let a potential problem be defined over the domain V with boundary S. The boundary conditions include Dirichlet conditions of the type $u(x)=u_0(x)$ over S_1 and Neumann conditions of the type $\partial u/\partial n(x)=q_0(x)$ over S_2 with $S=S_1 U S_2$, where $u_0(x)$, $q_0(x)$, as well as S_1 and S_2 may contain fuzzy information. In the BEM, nodes are introduced along the boundary S and the basic procedure applied to solve a problem using the BEM can be described by the following steps.

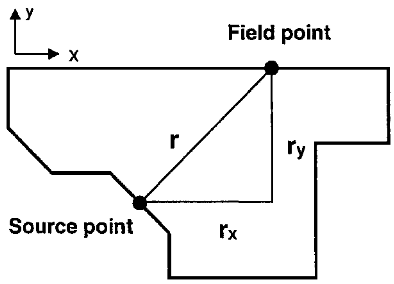

Fig. 3 Relationship between source point and field point.

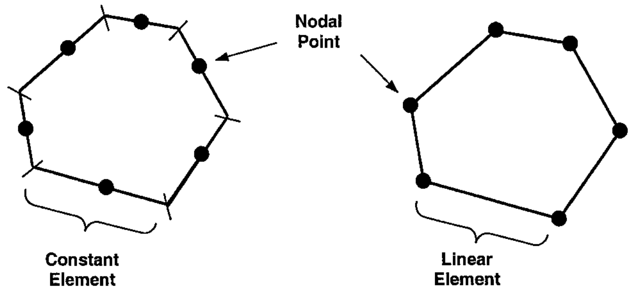

Fig. 4 Element meshes: constant and linear elements.

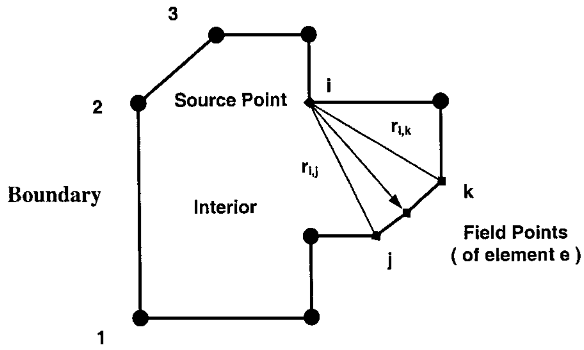

Fig. 5 Relationship of source point to field point of linear boundary element.

Governing Equation

The potential flow is governed by the Poisson equation:

$$k\nabla^2 u = b \tag{19}$$

where b is the internal source, k is the conductivity of the material, and u is the response function, such as temperature in a heat transfer problem. When the divergence theorem is applied, Eq. (19) can be used to obtain

$$u(p) + \int_S q^* u \, \mathrm{d}S = \int_S u^* q \, \mathrm{d}S \tag{20}$$

where $u(p)$ is the value of u at the source point p, u^* and q^* are the fundamental solutions that represent the potential and flux density, respectively, resulting from a point source in an infinite material, and S is the surface or boundary of the region. If r is the distance between the source point (an arbitrary point) and the field point (a point in an element on the boundary) with r_x and r_y denoting the components of r along the x and y directions (Fig. 3), the fundamental solutions for an isotropic material are given by

$$u^* = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right) \tag{21}$$

$$q^* = \frac{\partial u^*}{\partial n} = \frac{\partial u^* \partial r}{\partial r \partial n} = \frac{-1}{2\pi r^2}(r_x n_x + r_y n_y) \tag{22}$$

If the domain and the boundary are defined imprecisely, the radius vector r and the components r_x, r_y, n_x, and n_y are to be treated as fuzzy quantities.

Boundary Integral Equation

To take care of points p placed at different locations on the boundary, a multiplier $c(p)$ is introduced as

$$c(p) = \theta/2\pi \tag{23}$$

where θ is the angle subtended by the material around the source point and the resulting boundary integral equation is rewritten as

$$c(p)u(p) + \int_S q^* u \, \mathrm{d}S = \int_S u^* q \, \mathrm{d}S \tag{24}$$

Equation (23) indicates that $c(p)$ is equal to 0 if p is located completely outside the material and 0.5 if p is situated on a smooth boundary. If the domain is fuzzy, $c(p)$ appearing in Eqs. (23) and (24) is also to be treated as a fuzzy quantity. The integrals in Eq. (24) are to be evaluated using the fuzzy integration procedures indicated earlier.

Discretization

When the boundary is divided into elements (Fig. 4), the boundary integral equation can be written in a discrete form:

$$c(p)u(p) + \sum_e \int_{S^{(e)}} q^* u \, \mathrm{d}S = \sum_e \int_{S^{(e)}} u^* q \, \mathrm{d}S \tag{25}$$

where $S^{(e)}$ is the boundary of element e. The variations of u and q inside a typical element e are expressed, using suitable interpolation (shape) functions to describe the geometry and nodal values of potential and flux density over each element, as

$$u = \Phi^T \boldsymbol{u}^{(e)}, \qquad q = \Phi^T \boldsymbol{q}^{(e)} \tag{26}$$

where $\Phi^T = [\phi_1, \phi_2, \ldots, \phi_n]$ is the set of interpolation functions, $\boldsymbol{u}^{(e)^T} = [u_1, u_2, \ldots, u_n]$, $\boldsymbol{q}^{(e)^T} = [q_1, q_2, \ldots, q_n]$, and n is the number of nodes in element e. In view of Eq. (26), Eq. (25) can be expressed as

$$c(p)u(p) + \sum_e \int_{S^{(e)}} q^* \Phi^T \, \mathrm{d}S \cdot \boldsymbol{u}^{(e)} = \sum_e \int_{S^{(e)}} u^* \Phi^T \, \mathrm{d}S \cdot \boldsymbol{q}^{(e)} \tag{27}$$

For a linear element, the interpolation (shape) functions are defined in terms of the local coordinate ξ as

$$\phi_1 = \tfrac{1}{2}(1 - \xi), \qquad \phi_2 = \tfrac{1}{2}(1 + \xi) \tag{28}$$

where the nodes are placed on both ends of the element of length L_e and ϕ_1 and ϕ_2 represent their corresponding shape functions.

Matrix Equations

During numerical computation, the integral terms are to be evaluated for each combination of source and field points using their nodal information. For example, as shown in Fig. 5, if node i denotes the source point, its effect on element e, which is composed of nodes j and k, can be determined by expressing the integral terms of Eq. (27) as

$$\int_{S^{(e)}} q^* [\phi_1 \quad \phi_2] \, \mathrm{d}S \begin{Bmatrix} u_j \\ u_k \end{Bmatrix} = [\bar{h}_{i,j} \quad \bar{h}_{i,k}] \begin{Bmatrix} u_j \\ u_k \end{Bmatrix} = \bar{h}_{i,j} u_j + \bar{h}_{i,k} u_k \tag{29}$$

$$\int_{S^{(e)}} u^* [\phi_1 \quad \phi_2] \, \mathrm{d}S \begin{Bmatrix} q_j \\ q_k \end{Bmatrix} = [g_{i,j} \quad g_{i,k}] \begin{Bmatrix} q_j \\ q_k \end{Bmatrix} = g_{i,j} q_j + g_{i,k} q_k \tag{30}$$

Using p-point Gaussian integration, the coefficients $g_{i,j}$ and $\bar{h}_{i,j}$ can be expressed as

$$\bar{h}^a_{i,j} = -\frac{l_j}{2} \int_{-1}^{1} \frac{1}{2\pi r^2}(r_x n_x + r_y n_y) \frac{1}{2}(1 - \xi) \, \mathrm{d}\xi$$

$$= -\frac{l_j}{4\pi} \sum_1^p \omega_k (1 - \lambda_k) \frac{(r_x n_x + r_y n_y)}{\left(r_x^2 + r_y^2\right)} \tag{31}$$

$$g_{i,j}^{a} = -\frac{l_j}{2}\int_{-1}^{1}\frac{1}{2\pi}\ln(r)\frac{1}{2}(1-\xi)\,d\xi = -\frac{l_j}{4\pi}\sum_{1}^{p}\omega_k(1-\lambda_k)\ln\left(\sqrt{r_x^2+r_y^2}\right) \tag{32}$$

where ω_k are the weighting factors and a denotes the first node. When the second (end) node is denoted as b, the two integral terms will yield

$$\bar{h}_{i,j}^{b} = -\frac{l_j}{2}\int_{-1}^{1}\frac{1}{2\pi r^2}(r_x n_x + r_y n_y)\frac{1}{2}(1+\xi)\,d\xi = -\frac{l_j}{4\pi}\sum_{1}^{p}\omega_k\lambda_k\frac{(r_x n_x + r_y n_y)}{\left(r_x^2+r_y^2\right)} \tag{33}$$

$$g_{i,j}^{b} = -\frac{l_j}{2}\int_{-1}^{1}\frac{1}{2\pi}\ln(r)\frac{1}{2}(1+\xi)\,d\xi = -\frac{l_j}{4\pi}\sum_{1}^{p}\omega_k\lambda_k\ln\left(\sqrt{r_x^2+r_y^2}\right) \tag{34}$$

When the source and field points coincide ($i = j$), the coefficients $g_{i,j}$ and $\bar{h}_{i,j}$ are given by $\bar{h}_{i,i} = 0$ and

$$g_{ii} = -\frac{1}{2\pi}\left\{\frac{l_{i-1}}{2}\int_{-1}^{1}\ln\left[l_{i-1} - \frac{l_{i-1}(1+\xi)}{2}\right]\frac{1}{2}(1+\xi)\,d\xi + \frac{l_i}{2}\int_{-1}^{1}\ln\left[\frac{l_i(1+\xi)}{2}\right]\frac{1}{2}(1-\xi)\,d\xi\right\}$$
$$= -\frac{l_{i-1}}{4\pi}\left[\frac{3}{2} - \ln(l_{i-1})\right] - \frac{l_i}{4\pi}\left[\frac{3}{2} - \ln(l_i)\right] \tag{35}$$

By defining

$$h_{i,j} = \begin{cases}\bar{h}_{i,j} + c(p) & \text{if} \quad i = j\\ \bar{h}_{i,j} & \text{if} \quad i \neq j\end{cases} \tag{36}$$

Eq. (27) can be written as

$$\sum_e\left[\boldsymbol{H}^{(e)}\right]\cdot\boldsymbol{u}^{(e)} = \sum_e\left[\boldsymbol{G}^{(e)}\right]\cdot\boldsymbol{q}^{(e)} \tag{37}$$

By changing the source point from node i to another node, a different set of integral terms can be derived for each element on the boundary. If the total number of nodes in the model is N, the point source can be located at each of the N nodes in sequence, and N sets of equations can be derived. The final equations can be expressed in matrix form as

$$\underset{(N\times N)}{\boldsymbol{H}}\ \underset{(N\times 1)}{\boldsymbol{u}} = \underset{(N\times N)}{\boldsymbol{G}}\ \underset{(N\times 1)}{\boldsymbol{q}} \tag{38}$$

Equation (38) can be solved by incorporating the known boundary conditions.

Solution of Fuzzy Equations

Note that the matrices $\boldsymbol{H}$ and $\boldsymbol{G}$ are $N \times N$ fuzzy matrices and $\boldsymbol{u}$ and $\boldsymbol{q}$ are N-component fuzzy vectors. If the α-cuts approach is used, different sets of Eq. (38) will be valid at different α-cuts. Usually, N_1 fuzzy values of $\boldsymbol{u}$ and N_2 fuzzy values of $\boldsymbol{q}$ are known on S_1 and S_2, respectively, and hence, there are only N fuzzy unknowns in the system of Eq. (38). When the system is rearranged, the set of fuzzy equations to be solved can be expressed as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{39}$$

where $\boldsymbol{A} = [a_{ij}]$ and $\boldsymbol{b} = \{b_i\}$ denote the input fuzzy coefficient matrix and fuzzy right hand side vector, respectively, and $\boldsymbol{x} = \{x_i\}$ represents the fuzzy (output) solution vector. If fuzzy numbers are considered in terms of intervals of confidence at finite α levels, the problem of solving systems of fuzzy equations can be reduced to solving systems of interval equations.[30] The exact solutions to a system of interval equations, that is, the interval vector with the smallest radius, is called the hull of the solution set. The computation of the hull of the solution set, in general, is extremely difficult. According to Neumaier,[23] for dimensions N larger than about five, practical methods are available only in special cases. The necessary and sufficient conditions for the existence of a unique solution to a single fuzzy algebraic equation are given by Zhao and Govind.[31] For the fuzzy expression $\boldsymbol{Ax} = \boldsymbol{b}$, the relative spread of $\boldsymbol{b}$, defined as $(b^R/b^L)_\alpha$ for every level α, must be greater than or equal to the relative spread of $\boldsymbol{A}$ for a solution $\boldsymbol{x}$ to exist. The sufficient conditions state that, for a system of fuzzy equations $\boldsymbol{Ax} = \boldsymbol{b}$, the degree of uncertainty in a matrix $\boldsymbol{A}$ determines the minimum degree of uncertainty that must be present in the vector $\boldsymbol{b}$ for a solution to exist. Thus, unlike the case for real valued equations, there is a dependence between the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{b}$.

Several methods have been proposed in recent years for the solution of fuzzy equations $\boldsymbol{Ax} = \boldsymbol{b}$. At any particular α level, denoting the solution as $\{\underline{x}, \bar{x}\} \equiv \{(x_1^L, x_1^R), (x_2^L, x_2^R), \ldots, (x_N^L, x_N^R)\}$, an unconstrained minimization problem can be formulated to reduce the difference between the vectors $[\underline{A}, \bar{A}]\{\underline{x}, \bar{x}\}$ and $\{\underline{b}, \bar{b}\}$. A numerical method, based on the Taguchi approach, was presented by Rao and Chen[32] for the solution of Eq. (39). A two-level fractional factorial design of experiments based on orthogonal arrays was utilized to carry out the interval implementation. The Gaussian elimination method can be used by replacing crisp arithmetic operations by fuzzy relations. However, with interval arithmetic operations, the ranges of the elements of the solution vector $\boldsymbol{x}$ will be wider than the exact ranges. Thus, the solution might be incorrect, especially when large number of equations are involved. In this work, the fuzzy version of the Gaussian elimination method, coupled with a truncation procedure,[33] is used to solve Eq. (39). In the truncation method, the relative widths or spreads of the input parameters about their respective central value are noted. If the relative width of a computed response quantity about its central value is observed to be much larger than a maximum limit, specified based on the relative widths of the input parameters, the width of the computed response is truncated at the specified maximum limit. Let $[\underline{a}, \bar{a}]$ and $[\underline{b}, \bar{b}]$ be the input numbers and $[\underline{c}, \bar{c}]$ be the result of arithmetic operation. Corresponding to the central values of the input variables $a_0 = \frac{1}{2}(\underline{a}, \bar{a})$ and $b_0 = \frac{1}{2}(\underline{b}, \bar{b})$, let the crisp result be c_0. The maximum relative deviation of the input parameters t is given by

$$t = \max[(\bar{a} - \underline{a})/a_0, (\bar{b} - \underline{b})/b_0]$$

The deviations in the response are defined as

$$\Delta_1 = |(c_0 - \underline{c})/c_0|, \qquad \Delta_2 = |(\bar{c} - c_0)/c_0|$$

and total deviation is

$$\Delta = \Delta_1 + \Delta_2 = (\bar{c} - \underline{c})/c_0$$

When the maximum permissible relative deviation of c is defered as $2t$, the truncation procedure is implemented as follows[33]: 1) truncate $[\underline{c}, \bar{c}]$ as $[\underline{d}, \bar{d}]$ with $\underline{d} = \underline{c}$ and $\bar{d} = \bar{c}$ if $\Delta_1 \le t$ and $\Delta_2 \le t$, 2) $\underline{d} = c_0 + t(\underline{c} - c_0)$ and $\bar{d} = c_0 + t(\bar{c} - c_0)$ if $\Delta_1 > t$ and $\Delta_2 > t$, 3) $\underline{d} = \underline{c}$ and $\bar{d} = c_0 + t(\bar{c} - c_0)$ if $\Delta_1 \le t$ and $\Delta_2 > t$, and 4) $\bar{d} = \bar{c}$ and $\underline{d} = c_0 + t(\underline{c} - c_0)$ if $\Delta_1 > t$ and $\Delta_2 \le t$. This truncation approach, although approximate, was found to yield reasonably accurate results.

Numerical Example

To illustrate the BEM using fuzzy α-cuts analysis, a heat transfer problem (Fig. 6), is considered. For simplicity, only 12 boundary elements are used to find the temperatures (potentials) and flux densities over the four edges of the model. Four Gauss integration points are used for numerical evaluation of the coefficients $g_{i,j}$ and $\bar{h}_{i,j}$. The boundary element analysis is conducted using constant elements (each of length 1) by treating all parameters as crisp numbers (using the same upper and lower limits), and the resulting element temperatures and flux densities are shown in Table 2. The next analysis is

Table 2 Results of crisp analysis

Element number	Nodal position $X-Y$	Temperatures u	Flux densities q
1	(1.00, 1.00)–(0.00, 0.00)	(76.10454, 76.10454)	(0.00000, 0.00000)
2	(3.00, 3.00)–(0.00, 0.00)	(23.85633, 23.85633)	(0.00000, 0.00000)
3	(4.00, 4.00)–(1.00, 1.00)	(0.00000, 0.00000)	(−56.31438, −56.31438)
4	(3.00, 3.00)–(2.00, 2.00)	(23.85633, 23.85633)	(0.00000, 0.00000)
5	(1.00, 1.00)–(2.00, 2.00)	(76.10454, 76.10454)	(0.00000, 0.00000)
6	(0.00, 0.00)–(1.00, 1.00)	(100.00000, 100.00000)	(56.49581, 56.49581)

Table 3 Results with nodal coordinates as interval parameters

Element number	Nodal position $X-Y$	Temperatures u	Flux densities q
1	(0.49, 0.51)–(0.00, 0.00)	(73.91549, 78.27725)	(0.00000, 0.00000)
2	(1.49, 1.51)–(0.00, 0.00)	(22.31636, 25.43906)	(0.00000, 0.00000)
3	(1.98, 2.02)–(0.50, 0.50)	(0.00000, 0.00000)	(−59.86688, −53.10689)
4	(1.49, 1.51)–(1.00, 1.00)	(22.76732, 24.99040)	(0.00000, 0.00000)
5	(0.49, 0.51)–(1.00, 1.00)	(74.25630, 77.89499)	(0.00000, 0.00000)
6	(0.00, 0.00)–(0.50, 0.50)	(100.00000, 100.00000)	(55.05313, 58.11695)

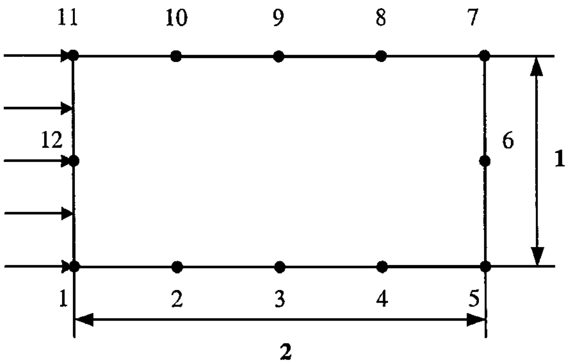

Fig. 6 Model of the heat transfer problem.

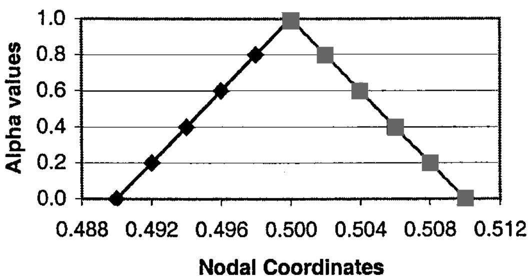

Fig. 7 Membership function of x coordinate of node 1: ◆, lower bounds and ■, upper bounds.

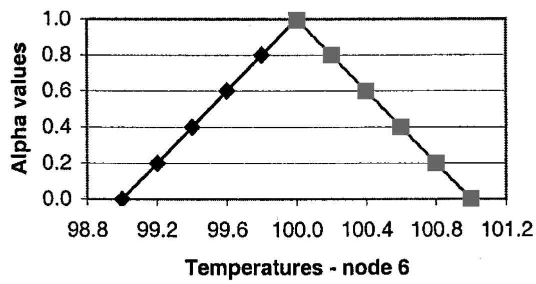

Fig. 8 Membership function of temperature of node 6: ◆, lower bounds and ■, upper bounds.

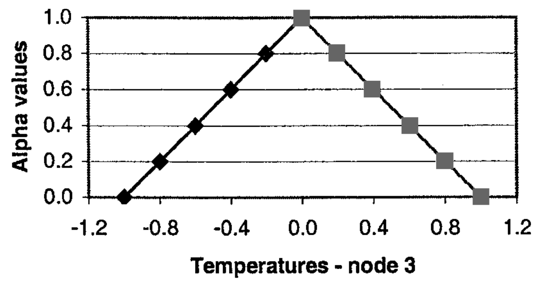

Fig. 9 Membership function of temperature of node 3: ◆, lower bounds and ■, upper bounds.

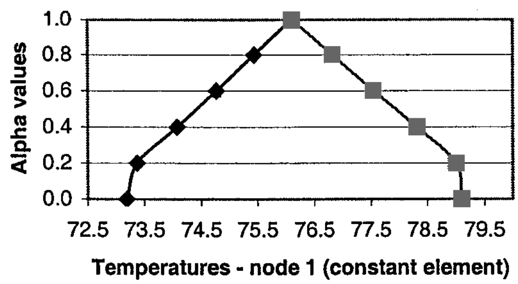

Fig. 10 Interval range, output data: temperatures node 1; ◆, lower bounds; and ■, upper bounds.

conducted by assuming the nodal coordinates as interval parameters. The input data used and the results (element temperatures and flux densities) are shown in Table 3.

The boundary element analysis with the fuzzy α-cuts method is conducted by treating both the specified temperatures and nodal coordinates as interval parameters. The membership functions of the input data, namely, the x coordinate of node 1 (midpoint of element 1) and temperatures of nodes 6 and 3 are shown in Figs. 7–9. For simplicity, linear memberships are assumed for the input parameters. Six discrete values of α are used in the numerical computations. The numerical results obtained with various α levels are given in Table 4. The membership functions of the resulting temperatures of nodes 1 and 4 (midpoints of elements 1 and 4) are shown in Figs. 10 and 11, respectively. It can be seen that, although the membership functions are not strictly linear, the behaviors can be seen to be similar to those of the input variables. Figures 12 and 13 show the solution for flux densities of nodes 6 and 3, which indicate that the interval ranges do not change in a regular manner. Observe that the exact (crisp) result (when $\alpha = 1$) is not included in the interval range corresponding to the first level (when $\alpha = 0$), although the interval range of the result is reduced with the increase of α level. Furthermore, the range of the parameter in each lower level continues to decrease instead of increase. Even when $\alpha = 0.8$, the interval range is found to be beyond the ranges corresponding to the lower levels, which means that we cannot expect the correct range by judging only from the solution of a lower α level.

Table 4 Results with nodal coordinates and specified temperatures as interval parameters

Element number	Nodal position X coordinate	Nodal position Y coordinate	Temperatures u	Flux densities q
			α Value 0.0	
1	(0.490, 0.510)	(0.000, 0.000)	(73.18640, 79.10325)	(0.00000, 0.00000)
2	(1.500, 1.500)	(0.000, 0.000)	(22.07362, 24.82181)	(0.00000, 0.00000)
3	(2.000, 2.000)	(0.490, 0.510)	(−1.00000, 1.00000)	(−55.04684, −54.22866)
4	(1.500, 1.500)	(1.000, 1.000)	(22.21874, 24.72392)	(0.00000, 0.00000)
5	(0.490, 0.510)	(1.000, 1.000)	(73.40370, 79.02527)	(0.00000, 0.00000)
6	(0.000, 0.000)	(0.490, 0.510)	(99.00000, 101.00000)	(57.68102, 61.26431)
			α Value 0.2	
1	(0.492, 0.508)	(0.000, 0.000)	(73.36914, 79.00032)	(0.00000, 0.00000)
2	(1.500, 1.500)	(0.000, 0.000)	(22.14524, 24.88318)	(0.00000, 0.00000)
3	(2.000, 2.000)	(0.492, 0.508)	(−0.80000, 0.80000)	(−58.08942, −51.75895)
4	(1.500, 1.500)	(1.000, 1.000)	(22.56064, 24.51249)	(0.00000, 0.00000)
5	(0.492, 0.508)	(1.000, 1.000)	(73.94673, 78.42147)	(0.00000, 0.00000)
6	(0.000, 0.000)	(0.492, 0.508)	(99.20000, 100.80000)	(57.66912, 60.23200)
			α Value 0.4	
1	(0.494, 0.506)	(0.000, 0.000)	(74.07705, 78.31200)	(0.00000, 0.00000)
2	(1.500, 1.500)	(0.000, 0.000)	(22.60026, 24.58305)	(0.00000, 0.00000)
3	(2.000, 2.000)	(0.494, 0.506)	(−0.60000, 0.60000)	(−57.52501, −52.93470)
4	(1.500, 1.500)	(1.000, 1.000)	(22.89510, 24.31995)	(0.00000, 0.00000)
5	(0.494, 0.506)	(1.000, 1.000)	(74.48785, 77.82788)	(0.00000, 0.00000)
6	(0.000, 0.000)	(0.494, 0.506)	(99.40000, 100.80000)	(57.54393, 59.24036)
			α Value 0.6	
1	(0.496, 0.504)	(0.000, 0.000)	(74.76868, 77.54880)	(0.00000, 0.00000)
2	(1.500, 1.500)	(0.000, 0.000)	(23.03675, 24.31196)	(0.00000, 0.00000)
3	(2.000, 2.000)	(0.496, 0.504)	(−0.40000, 0.40000)	(−57.04119, −54.08512)
4	(1.500, 1.500)	(1.000, 1.000)	(23.22235, 24.14634)	(0.00000, 0.00000)
5	(0.496, 0.504)	(1.000, 1.000)	(75.02753, 77.24410)	(0.00000, 0.00000)
6	(0.000, 0.000)	(0.496, 0.504)	(99.60000, 100.40000)	(57.30646, 58.28818)
			α Value 0.8	
1	(0.498, 0.502)	(0.000, 0.000)	(75.43708, 76.82105)	(0.00000, 0.00000)
2	(1.500, 1.500)	(0.000, 0.000)	(23.53426, 24.16714)	(0.00000, 0.00000)
3	(2.000, 2.000)	(0.498, 0.502)	(−0.20000, 0.20000)	(−57.06720, −55.59831)
4	(1.500, 1.500)	(1.000, 1.000)	(23.62652, 24.08436)	(0.00000, 0.00000)
5	(0.498, 0.502)	(1.000, 1.000)	(75.56638, 76.67032)	(0.00000, 0.00000)
6	(0.000, 0.000)	(0.498, 0.502)	(99.80000, 100.20000)	(56.21524, 56.70574)
			α Value 1.0	
1	(0.500, 0.500)	(0.000, 0.000)	(76.10454, 76.10454)	(0.00000, 0.00000)
2	(1.500, 1.500)	(0.000, 0.000)	(23.85633, 23.85633)	(0.00000, 0.00000)
3	(2.000, 2.000)	(0.500, 0.500)	(0.00000, 0.00000)	(−56.31438, −56.31438)
4	(1.500, 1.500)	(1.000, 1.000)	(23.85633, 23.85633)	(0.00000, 0.00000)
5	(0.500, 0.500)	(1.000, 1.000)	(76.10454, 76.10454)	(0.00000, 0.00000)
6	(0.000, 0.000)	(0.500, 0.500)	(100.00000, 100.00000)	(56.49581, 56.49581)

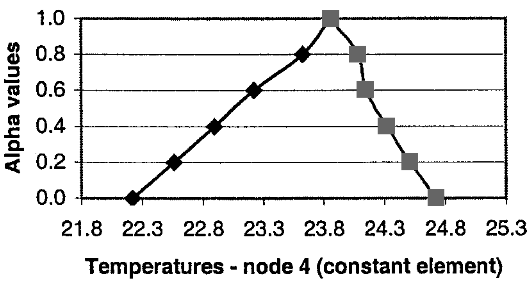

Fig. 11 Interval range, output data: temperatures, node 4; ◆, lower bounds; and ■, upper bounds.

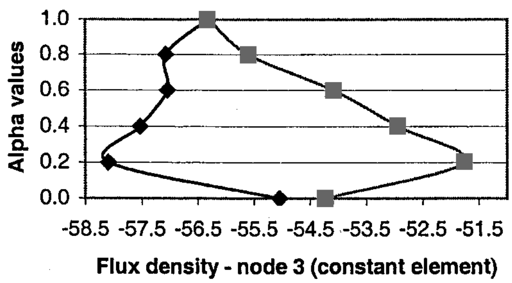

Fig. 13 Interval range, output data: flux density, node 3; ◆, lower bounds; and ■, upper bounds.

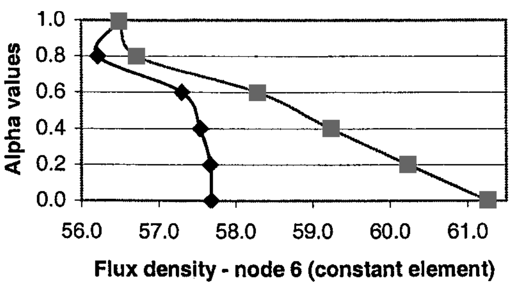

Fig. 12 Interval range, output data: flux density, node 6; ◆, lower bounds; and ■, upper bounds.

A possible reason or hypothesis that led to this result is the use of Gauss elimination for solving matrix interval equations. During Gauss elimination, the elements of equations related to the flux density are very sensitive to the computational operations; small changes in certain elements lead to large differences in the results. The small tolerances (ranges) considered as interval values reflect this kind of possibility. To verify the hypothesis, the Gauss elimination is used, with linear elements, to perform crisp calculations instead of interval analysis, taking the lower and upper bounds of interval values, respectively, as the input data, and the results obtained are indicated in Figs. 14–17. Comparing these solutions with the corresponding results shown in Figs. 10–13, it can be observed that the results have been improved substantially.

Table 5 Results with linear elements and nodal coordinates as interval parameters

Element no. (nodes)	First nodal position $X-Y$	Temperatures u	Flux densities q
01 (01−02)	(0.00, 0.00)−(0.00, 0.00)	(100.00000, 100.00000)	(21.34625, 24.40519)
02 (02−03)	(0.49, 0.51)−(0.00, 0.00)	(73.46811, 78.77365)	(0.00000, 0.00000)
03 (03−04)	(1.00, 1.00)−(0.00, 0.00)	(48.80527, 51.28352)	(0.00000, 0.00000)
04 (04−05)	(1.49, 1.51)−(0.00, 0.00)	(03.15515, 24.69183)	(0.00000, 0.00000)
05 (05−06)	(2.00, 2.00)−(0.00, 0.00)	(0.00000, 0.00000)	(−26.99403, −19.99784)
06 (06−07)	(2.00, 2.00)−(0.49, 0.51)	(0.00000, 0.00000)	(−58.34694, −55.72943)
07 (07−08)	(2.00, 2.00)−(1.00, 1.00)	(0.00000, 0.00000)	(−26.90365, −20.1080)
08 (08−09)	(1.49, 1.51)−(1.00, 1.00)	(23.13027, 24.59735)	(0.00000, 0.00000)
09 (09−10)	(1.00, 1.00)−(1.00, 1.00)	(48.84114, 51.24222)	(0.00000, 0.00000)
10 (10−11)	(0.49, 0.51)−(1.00, 1.00)	(73.16183, 79.28561)	(0.00000, 0.00000)
11 (11−12)	(0.00, 0.00)−(1.00, 1.00)	(100.00000, 100.00000)	(21.44920, 24.36966)
12 (12−13)	(0.00, 0.00)−(0.49, 0.51)	(100.00000, 100.00000)	(53.24174, 59.70083)

Table 6 Results with linear elements and nodal coordinates and specified temperatures as interval parameters

Element no. (nodes)	First nodal position $X-Y$	Temperatures u	Flux densities q
		α Value 0.0	
01 (01−02)	(0.000, 0.000)−(0.000, 0.000)	(99.00000, 101.00000)	(15.19668, 28.71217)
02 (02−03)	(0.490, 0.510)−(0.000, 0.000)	(72.54857, 79.97842)	(0.00000, 0.00000)
03 (03−04)	(1.000, 1.000)−(0.000, 0.000)	(47.29413, 53.10330)	(0.00000, 0.00000)
04 (04−05)	(1.490, 1.510)−(0.000, 0.000)	(21.84369, 26.24462)	(0.00000, 0.00000)
05 (05−06)	(2.000, 2.000)−(0.000, 0.000)	(−1.00000, 1.00000)	(−32.77940, −16.06975)
06 (06−07)	(2.000, 2.000)−(0.490, 0.510)	(−1.00000, 1.00000)	(−63.21011, −51.86862)
07 (07−08)	(2.000, 2.000)−(1.000, 1.000)	(−1.00000, 1.00000)	(−32.74636, −16.10911)
08 (08−09)	(1.490, 1.510)−(1.000, 1.000)	(21.83028, 26.14464)	(0.00000, 0.00000)
09 (09−10)	(1.000, 1.000)−(1.000, 1.000)	(47.33029, 53.05787)	(0.00000, 0.00000)
10 (10−11)	(0.490, 0.510)−(1.000, 1.000)	(72.26024, 80.50919)	(0.00000, 0.00000)
11 (11−12)	(0.000, 0.000)−(1.000, 1.000)	(99.00000, 101.00000)	(15.31628, 28.68861)
12 (12−13)	(0.000, 0.000)−(0.490, 0.510)	(99.00000, 101.00000)	(46.38347, 65.35538)
		α Value 0.6	
01 (01−02)	(0.000, 0.000)−(0.000, 0.000)	(99.00000, 100.40000)	(20.48084, 25.50030)
02 (02−03)	(0.496, 0.504)−(0.000, 0.000)	(74.62815, 77.55087)	(0.00000, 0.00000)
03 (03−04)	(1.000, 1.000)−(0.000, 0.000)	(48.88970, 51.15074)	(0.00000, 0.00000)
04 (04−05)	(1.496, 1.504)−(0.000, 0.000)	(23.06177, 24.78211)	(0.00000, 0.00000)
05 (05−06)	(2.000, 2.000)−(0.000, 0.000)	(−0.40000, 0.40000)	(−26.67859, −19.94060)
06 (06−07)	(2.000, 2.000)−(0.496, 0.504)	(−0.40000, 0.40000)	(−58.64665, −55.26485)
07 (07−08)	(2.000, 2.000)−(1.000, 1.000)	(0.400000, 0.40000)	(−26.61879, −20.00961)
08 (08−09)	(1.496, 1.504)−(1.000, 1.000)	(23.04850, 24.74607)	(0.00000, 0.00000)
09 (09−10)	(1.000, 1.000)−(1.000, 1.000)	(48.90413, 51.13567)	(0.00000, 0.00000)
10 (10−11)	(0.496, 0.504)−(1.000, 1.000)	(74.49502, 77.74660)	(0.00000, 0.00000)
11 (11−12)	(0.000, 0.000)−(1.000, 1.000)	(99.60000, 100.40000)	(20.51836, 25.48287)
12 (12−13)	(0.000, 0.000)−(0.496, 0.504)	(99.60000, 100.40000)	(53.09150, 60.51192)
		α Value 1.0	
01 (01−02)	(0.000, 0.000)−(0.000, 0.000)	(100.00000, 100.00000)	(23.15176, 23.15176)
02 (02−03)	(0.500, 0.500)−(0.000, 0.000)	(76.08048, 76.08048)	(0.00000, 0.00000)
03 (03−04)	(1.000, 1.000)−(0.000, 0.000)	(49.99986, 49.99986)	(0.00000, 0.00000)
04 (04−05)	(1.500, 1.500)−(0.000, 0.000)	(23.91960, 23.91960)	(0.00000, 0.00000)
05 (05−06)	(2.000, 2.000)−(0.000, 0.000)	(−0.00000, 0.00000)	(−23.15139, −23.15139)
06 (06−07)	(2.000, 2.000)−(0.500, 0.500)	(−0.00000, 0.00000)	(−56.89830, −56.89830)
07 (07−08)	(2.000, 2.000)−(1.000, 1.000)	(0.000000, 0.00000)	(−23.15138, −23.15138)
08 (08−09)	(1.500, 1.500)−(1.000, 1.000)	(23.91957, 23.91957)	(0.00000, 0.00000)
09 (09−10)	(1.000, 1.000)−(1.000, 1.000)	(49.99998, 49.99998)	(0.00000, 0.00000)
10 (10−11)	(0.500, 0.500)−(1.000, 1.000)	(76.08045, 76.08045)	(0.00000, 0.00000)
11 (11−12)	(0.000, 0.000)−(1.000, 1.000)	(100.00000, 100.00000)	(23.15178, 23.15178)
12 (12−13)	(0.000, 0.000)−(0.500, 0.500)	(100.00000, 100.00000)	(56.89667, 56.89667)

The analysis is conducted by using linear element (each of length 1) and by treating only the nodal coordinates as interval input parameters. The specified nodal temperatures are chosen as crisp values (with the same lower and upper bound values). The results of analysis are shown in Table 5. Next, the nodal coordinates and the specified nodal temperatures are assumed to be interval numbers, and the results of analysis are shown in Table 6 for different values of α. As before, linear membership functions are assumed for all input parameters. The membership functions of the temperatures of nodes 2 and 8 as well as those of the flux densities of nodes 12 and 6 are shown in Figs. 14–17. Figure 14 can be compared with Fig. 10 because, in the case of linear elements, the position of node 2 is assumed to be same as the position of node 1 in the case of constant elements. For similar reasons, Figs. 15–17 can be compared with Figs. 11–13, respectively.

It can be seen that the distributions of interval ranges for both the temperatures and flux densities have been improved compared to the constant elements. The results shown in Figs. 14–17 can be seen to be compatible with the linear interval ranges of the input data. Finally, the analysis is conducted by choosing all of the lower bounds of the input nodal positions, temperatures, and flux densities as crisp (deterministic) data. Similarly, the analysis is made by using all of the upper bounds of the input parameters as crisp data. These two types of analyses are conducted at each discrete α value separately, and the results obtained are indicated in Table 7. As can be seen, this procedure does not involve the interval analysis.

In Table 7, the interval ranges of the output parameters can be seen to be smaller than those listed in Table 6. This implies that when interval method is used, the interval ranges of the output parameters have been increased due to the interval arithmetic process (for the

Table 7 Results with lower and upper bounds of input parameters as crisp data

Element no. (nodes)	First nodal position $X-Y$	Temperatures u	Flux densities q
		α Value 0.0	
01 (01–02)	(0.000, 0.000)–(0.000, 0.000)	(99.00000, 101.00000)	(23.06016, 23.24290)
02 (02–03)	(0.490, 0.510)–(0.000, 0.000)	(75.55788, 76.60304)	(0.00000, 0.00000)
03 (03–04)	(1.000, 1.000)–(0.000, 0.000)	(48.96932, 51.03073)	(0.00000, 0.00000)
04 (04–05)	(1.490, 1.510)–(0.000, 0.000)	(23.39965, 24.44026)	(0.00000, 0.00000)
05 (05–06)	(2.000, 2.000)–(0.000, 0.000)	(−1.00000, 1.00000)	(−22.99649, −23.30833)
06 (06–07)	(2.000, 2.000)–(0.490, 0.510)	(−1.00000, 1.00000)	(−56.91661, −56.87681)
07 (07–08)	(2.000, 2.000)–(1.000, 1.000)	(−1.00000, 1.00000)	(−23.24250, −23.05981)
08 (08–09)	(1.490, 1.510)–(1.000, 1.000)	(23.39700, 24.44218)	(0.00000, 0.00000)
09 (09–10)	(1.000, 1.000)–(1.000, 1.000)	(48.96911, 51.03053)	(0.00000, 0.00000)
10 (10–11)	(0.490, 0.510)–(1.000, 1.000)	(75.55982, 76.60038)	(0.00000, 0.00000)
11 (11–12)	(0.000, 0.000)–(1.000, 1.000)	(99.00000, 101.00000)	(22.99690, 23.30868)
12 (12–13)	(0.000, 0.000)–(0.490, 0.510)	(99.00000, 101.00000)	(56.87512, 56.91502)
		α Value 0.6	
01 (01–02)	(0.000, 0.000)–(0.000, 0.000)	(99.60000, 100.40000)	(23.11518, 23.18828)
02 (02–03)	(0.496, 0.504)–(0.000, 0.000)	(75.87142, 76.28948)	(0.00000, 0.00000)
03 (03–04)	(1.000, 1.000)–(0.000, 0.000)	(49.58766, 50.41222)	(0.00000, 0.00000)
04 (04–05)	(1.496, 1.504)–(0.000, 0.000)	(23.71154, 24.12778)	(0.00000, 0.00000)
05 (05–06)	(2.000, 2.000)–(0.000, 0.000)	(−0.40000, 0.40000)	(−23.21390, −23.08919)
06 (06–07)	(2.000, 2.000)–(0.496, 0.504)	(−0.40000, 0.40000)	(−56.90601, −56.89009)
07 (07–08)	(2.000, 2.000)–(1.000, 1.000)	(0.40000, 0.40000)	(−23.18789, −23.11481)
08 (08–09)	(1.496, 1.504)–(1.000, 1.000)	(23.71056, 24.12864)	(0.00000, 0.00000)
09 (09–10)	(1.000, 1.000)–(1.000, 1.000)	(49.58762, 50.41219)	(0.00000, 0.00000)
10 (10–11)	(0.496, 0.504)–(1.000, 1.000)	(75.87228, 76.28850)	(0.00000, 0.00000)
11 (11–12)	(0.000, 0.000)–(1.000, 1.000)	(99.60000, 100.40000)	(23.08959, 23.21427)
12 (12–13)	(0.000, 0.000)–(0.496, 0.504)	(99.60000, 100.40000)	(56.88844, 56.90441)
		α Value 1.0	
01 (01–02)	(0.000, 0.000)–(0.000, 0.000)	(100.00000, 100.00000)	(23.15177, 23.15177)
02 (02–03)	(0.500, 0.500)–(0.000, 0.000)	(76.08045, 76.08045)	(0.00000, 0.00000)
03 (03–04)	(1.000, 1.000)–(0.000, 0.000)	(49.99992, 49.99992)	(0.00000, 0.00000)
04 (04–05)	(1.500, 1.500)–(0.000, 0.000)	(23.91960, 23.91960)	(0.00000, 0.00000)
05 (05–06)	(2.000, 2.000)–(0.000, 0.000)	(0.00000, 0.00000)	(−23.15138, −23.15138)
06 (06–07)	(2.000, 2.000)–(0.500, 0.500)	(0.00000, 0.00000)	(−56.89831, −56.89831)
07 (07–08)	(2.000, 2.000)–(1.000, 1.000)	(0.00000, 0.00000)	(−23.15138, −23.15138)
08 (08–09)	(1.500, 1.500)–(1.000, 1.000)	(23.91960, 23.91960)	(0.00000, 0.00000)
09 (09–10)	(1.000, 1.000)–(1.000, 1.000)	(49.99992, 49.99992)	(0.00000, 0.00000)
10 (10–11)	(0.500, 0.500)–(1.000, 1.000)	(76.08045, 76.08045)	(0.00000, 0.00000)
11 (11–12)	(0.000, 0.000)–(1.000, 1.000)	(100.00000, 100.00000)	(23.15177, 23.15177)
12 (12–13)	(0.000, 0.000)–(0.500, 0.500)	(100.00000, 100.00000)	(56.89668, 56.89668)

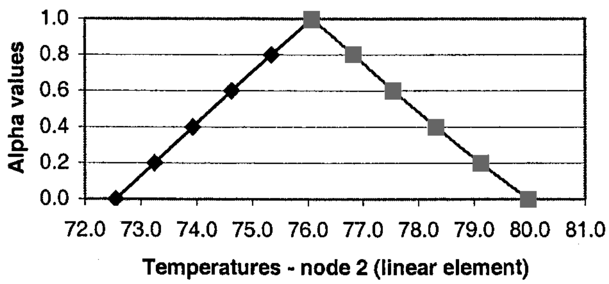

Fig. 14 Interval range, output data: temperatures, node 2; ◆, lower bounds; and ■, upper bounds.

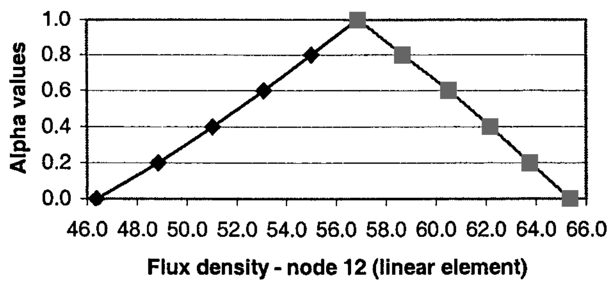

Fig. 16 Interval range, output data: flux density, node 12; ◆, lower bounds; and ■, upper bounds.

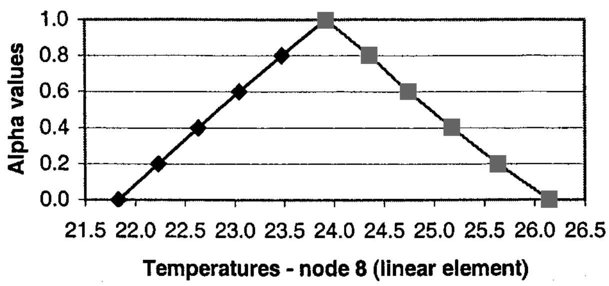

Fig. 15 Interval range, output data: temperatures, node 8; ◆, lower bounds; and ■, upper bounds.

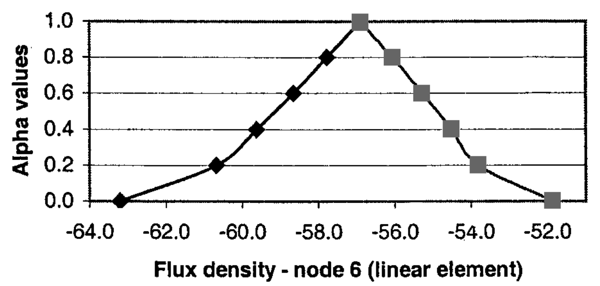

Fig. 17 Interval range, output data: flux density, node 6; ◆, lower bounds; and ■, upper bounds.

same interval ranges of the input data). Note that we cannot simply take the lower and upper bound values of the output parameters to represent the lower and upper bounds on the solution (output) just because we used lower and upper bounds of the input data in the calculations. The real lower and upper limits on a response parameter can only be established by conducting a combinatorial analysis. The results given by the interval analysis (Table 6) have been found to represent more realistic interval ranges of the output parameters for different values of α.

Conclusions

A methodology is presented for the fuzzy boundary element analysis of imprecisely defined systems. The feasibility of the method is demonstrated through a simple potential flow problem. The extension of the methodology for the solution of complex engineering analysis problems, including structural applications, is currently under investigation. This methodology permits the use of a new philosophy for the solution of engineering analysis problems involving imprecisely defined geometry, external actions (loads), material properties, and boundary conditions. When membership functions are constructed for the imprecise quantities, the fuzzy calculus and integration techniques are used to derive the boundary element equations. The resulting fuzzy system of equations are solved using the theory of interval equations. It is established that a fuzzy number in $\Re$ does not have an additive inverse, and the operation of fuzzy multiplication is, in general, not distributive. Hence, classical solution techniques, such as Gaussian elimination, cannot be extended to fuzzy equations directly. In this work, the fuzzified version of Gaussian elimination, with a truncation scheme, is used to obtain reasonably accurate response parameters. For larger boundary element models, solution methods that are more robust and efficient will be required. An alternative to the truncation method is to use a scheme that considers the interaction of fuzzy numbers as outlined by Bonarini and Bontempi for the solution of differential equations.[34]

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S. Saigal
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